

## THE CANONICAL 2-GERBE OF A COMPLEX MANIFOLD

MARKUS UPMEIER

ABSTRACT. We present the construction of a holomorphic bundle 2-gerbe for each complex manifold  $X$ , a higher analog of the canonical line bundle. It is a geometric representative of the second Beilinson-Chern class. Also, an Atiyah class for gerbes is introduced and a Koszul-Malgrange type theorem is proven.

A fundamental object associated to complex manifolds  $X$  is its canonical line bundle  $\Lambda^{\text{top}}(T^*X)$ . The present paper deals with an extension of this concept. To explain the analogy, recall that the canonical bundle may be viewed as a representative of the first Chern class. In Theorem 3 we construct a geometric representative of the second Beilinson-Chern class, introduced in [Bei84]. This is a 2-gerbe and may be computed from the eigenvalues and eigenspaces of chart transition functions, just like the canonical bundle depends only on the product of these eigenvalues.

We are motivated by the relationship between 2-gerbes and line bundles on the space of curves, via transgression [CJM<sup>+</sup>05, Bry99b, Bry08, Wal12]. According to the paradigm of string topology, structure on the infinite-dimensional loop space may be studied through higher-categorical, finite-dimensional structure on the original space. This is the viewpoint of the present paper. Having applications to complex geometry in mind, we shall be particularly concerned with the holomorphic structure on our gerbe.

The 2-gerbe will be constructed within the framework of bundle gerbes, introduced in [Mur96] and further studied in [MS00, Mur10, Ste04].

To set this work apart from related results, we now briefly review the existing literature. In [BM96, BM94, Bry08] tautological gerbes are defined as Dixmier-Douady sheaves of groupoids, but the constructions involve infinite-dimensional spaces. This makes them less useful from the point of view of string topology. We shall construct instead a very explicit and finite-dimensional model. Moreover, we have *smooth* gerbes over compact, simple, simply-connected Lie groups [Mei03], certain quotients therefore [GR04a], for  $\text{SU}(n)$  [GR02], and complex reductive Lie groups [Bry]. Of course,  $\text{GL}(n, \mathbb{C})$  is of none of these types. For compact semi-simple Lie groups  $G$ , a smooth multiplicative structure on the tautological gerbe on  $G$  was constructed by cohomological arguments in [CJM<sup>+</sup>05] and [Wal10].

We now state our main results and give an overview of the present paper.

We begin in Section 1 by developing further the theory of holomorphic bundle gerbes started in [MS03, Section 7]. In 1.2 we study the relationship to smooth gerbes and present an analogue of the Koszul-Malgrange Theorem [KM58]. For holomorphic line bundles, the Atiyah class is the obstruction against holomorphic connections. In 1.4 we demonstrate how this idea extends to gerbes. The rest of the section reviews well-known terminology for smooth gerbes [Mur96, MS00, Mur10, Wal07] in our holomorphic context.

Our first theorem gives a very explicit and finite-dimensional construction of a holomorphic gerbe  $\mathcal{G}_{\text{can}}$  that is canonical in the sense that it belongs to the generator of  $H^2(\text{GL}(n, \mathbb{C}); \mathcal{O}^\times) = \mathbb{Z}$ . The construction of  $\mathcal{G}_{\text{can}}$  in Section 2.1 is an extension of the work [MS08] for the smooth Lie group  $\text{U}(n)$  to the non-compact group  $\text{GL}(n, \mathbb{C})$  and the holomorphic category. In Section 2.2 we then prove:

**Theorem 1.**  $\mathcal{G}_{\text{can}} = (\pi: Y \rightarrow \text{GL}_n(\mathbb{C}), L, m)$  defines a holomorphic gerbe. Its Dixmier-Douady class is the generator of  $H^2(\text{GL}(n, \mathbb{C}), \mathcal{O}^\times)$ .

The complex Lie group  $\text{GL}(n, \mathbb{C})$  is a Stein group. Using cohomological arguments developed in Section 3 for Stein manifolds, we prove in Section 4 that the existence of a multiplicative structure is a purely topological problem:

**Theorem 2.** Let  $G$  be a Stein group and let  $\mathcal{G}$  be a holomorphic gerbe on  $G$  with holomorphic connection. Then  $\mathcal{G}$  admits a holomorphic multiplicative structure with connection precisely when the topological Dixmier-Douady class  $\text{DD}(\mathcal{G}) \in H^3(G; \mathbb{Z})$  is in the image of the transgression map  $H^4(BG; \mathbb{Z}) \rightarrow H^3(G; \mathbb{Z})$ .

In particular, every holomorphic gerbe with connection on  $\text{GL}(n, \mathbb{C})$  admits a multiplicative structure (Corollary 23). Following the proof, we also give a more explicit description. It is special to our treatment that we use Stein spaces to study the Beilinson-Chern classes, instead of Deligne's theory of mixed Hodge structures [Del74] used in [Beĭ84, EV88]. Sections 5.1 and 5.2 introduce a notion of 2-gerbe, which is slightly weaker than that in [Ste04], and their Dixmier-Douady class. For each holomorphic vector bundles  $E \rightarrow X$  we present in Section 5.3 a construction for a 2-gerbe  $\mathfrak{G}(E)$ . We show:

**Theorem 3.** The associated 2-gerbe has the following properties:

- (1)  $\mathfrak{G}(f^*E) \cong f^*\mathfrak{G}(E)$  (functorial)
- (2) The topological Dixmier-Douady class is  $c_2(E)$ .
- (3) For  $X$  algebraic,  $\text{DD}(\mathfrak{G}(E)) = c_2^B(E)$  is the Beilinson-Chern class of  $E$ .

Applied to  $E = T^*X$  we obtain the *canonical 2-gerbe* of a complex manifold  $X$ .

**Notation.** For iterated fiber products we use the notation  $Y^{[i]} = Y \times_X \cdots \times_X Y$  and similarly for maps over  $X$ . By  $\text{pr}_{ijk} \dots$  we mean the projections onto the indicated factors. The sheaf of holomorphic  $k$ -forms is  $\Omega^k$  and  $\mathcal{O}^*$  is the sheaf of nowhere vanishing holomorphic functions. Unless stated otherwise, *all bundles and bundle maps below are assumed to be holomorphic*.

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## 1. HOLOMORPHIC BUNDLE GERBES AND CONNECTIONS

In this section we review well-established concepts for smooth gerbes in the holomorphic context, see [MS03]. Also some new results are developed, such a Koszul-Malgrange Theorem for gerbes in 1.2 or a generalization of the Atiyah class in 1.4. Note that Brylinski's holomorphic gerbes [Bry08], sheaves of groupoids with band  $\mathcal{O}^\times$ , are equivalent to the holomorphic bundle gerbes considered here.

**1.1. Holomorphic Gerbes.** Ordinary gerbes are geometric representatives of cohomology classes  $H^2(X, \mathbb{C}^*) \cong H^3(X; \mathbb{Z})$ . In the simplest approach, one uses Čech cocycles for an open cover of  $X$ , leading to Hitchin-Chatterjee gerbes [Cha98]. Below, we shall define holomorphic gerbes on complex manifolds  $X$ . These objects are designed to represent cohomology classes in  $H^2(X, \mathcal{O}_X^*)$ . We will adopt the ‘bundle gerbe’ approach from [Mur96] whose main advantage is the use of descent theory to avoid artificial choices of an open cover.

**Definition 4.** A *holomorphic gerbe*  $\mathcal{G}$  on  $X$  is a triple  $(\pi, L, m)$  consisting of a holomorphic submersion  $\pi: Y \rightarrow X$ , holomorphic line bundle  $L \rightarrow Y^{[2]}$ , and bundle isomorphism  $m: \text{pr}_{12}^* L \otimes \text{pr}_{23}^* L \rightarrow \text{pr}_{13}^* L$ . Denoting by  $L_{y_1, y_2}$  the fibers of  $L$ , we get maps  $m: L_{y_1, y_2} \otimes L_{y_2, y_3} \rightarrow L_{y_1, y_3}$ , called *multiplication*, which are required to be associative (see [Mur10, Definition 4.1]).

**Definition 5.** A *connection* on  $\mathcal{G} = (Y, L, m)$  is a holomorphic connection on  $L$  (see [Huy05, Definition 4.2.17]) preserved by the multiplication  $m$ . A *curving* on a gerbe with connection is a form  $f \in \Omega^2(Y)$  related to the curvature of  $L$  by

$$\text{pr}_2^* f - \text{pr}_1^* f = R(\nabla^L). \quad (1)$$

The *3-curvature* is then the unique  $\rho = R(\mathcal{G}, \nabla^L, f) \in \Omega^3(X)$  with  $\pi^* \rho = df$ .

There is also the weaker notion of *compatible connection* on  $\mathcal{G}$ . It consists of a connection on  $L$  only *compatible* with the holomorphic structure (see [Huy05]). The curving may then be a form of type  $(2, 0) + (1, 1)$ .

**1.2. Relationship to Smooth Gerbes.** Holomorphic gerbes *with* holomorphic connections may equivalently be described by connective data with certain properties. There is an obvious variant of Definition 4 in the category of smooth manifolds.

**Theorem 6.** Let  $\mathcal{G} = (\pi: Y \rightarrow X, L, m)$  be a smooth gerbe whose projection  $\pi$  is a holomorphic submersion. Then the following are equivalent:

- (1)  $\mathcal{G}$  is a holomorphic gerbe with holomorphic connection and curving  $f$ .
- (2)  $\mathcal{G}$  is a smooth gerbe with smooth connection and curving  $f$  of type  $(2, 0)$  and 3-curvature  $\rho$  of type  $(3, 0)$ .

*Proof.* It is obvious that a holomorphic connection has the properties stated in (2). Conversely, the  $(0, 1)$ -part  $(\nabla^L)^{(0,1)}$  of the smooth connection determines a Cauchy-Riemann operator on  $L$ . By equation (1), the curvature of the connection is of type  $(2, 0)$  which shows in particular that  $(\nabla^L)^{(0,1)} \circ (\nabla^L)^{(0,1)} = 0$  is flat. Therefore the Theorem of Koszul-Malgrange [KM58] shows that this Cauchy-Riemann operator defines a holomorphic structure on  $L$ . The multiplication  $m$  preserves (the  $(0, 1)$ -part of) the connection and is therefore holomorphic. Since the 3-curvature  $\rho$  of type  $(3, 0)$  satisfies  $\pi^* \rho = df$ , we have  $\bar{\partial} f = 0$ , so  $f$  is holomorphic.  $\square$

A similar statement applies to holomorphic gerbes with connection, but without curving (the hypothesis is then that  $R(\nabla^L)$  is of type  $(2, 0)$ ). Moreover, a compatible connection with curving on  $\mathcal{G}$  corresponds to a smooth connection whose curving has type  $(2, 0) + (1, 1)$ .

**1.3. Class in Deligne Cohomology.** Recall that the *Deligne complex*  $\mathbb{Z}(p)_D$  is the following complex of sheaves (see [Bry08, Section 1.5])

$$\mathbb{Z}(p)_D: 0 \longrightarrow \mathcal{O}^* \xrightarrow{d \log} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-1}.$$

The *Deligne cohomology* groups  $H^q(X, \mathbb{Z}(p)_D)$  are defined as the hypercohomology groups of this complex. For example,  $H^2(X, \mathbb{Z}(2)_D)$  is the group of holomorphic line bundles on  $X$  with holomorphic connection, see [Del91]. We now extend this classification to holomorphic gerbes  $\mathcal{G}$ , leading to the Dixmier-Douady class.

**Lemma 7** ([Lur09, Lemma 7.2.3.5]). *Let  $\mathcal{U} = \{U_\alpha\}$  be an open covering of a paracompact space  $X$ ,  $k \in \mathbb{N}$ . Given an open cover  $\mathcal{V}^{\alpha_0 \dots \alpha_k}$  of each  $(k+1)$ -fold intersection  $U_{\alpha_0 \dots \alpha_k}$ , there exists a refinement  $\{W_\beta\}$  of  $\mathcal{U}$  with the property that each  $(k+1)$ -fold intersection  $W_{\beta_0 \dots \beta_k}$  is a subset of some element of  $\mathcal{V}^{\alpha_0 \dots \alpha_k}$ .*

**Corollary 8.** *Let  $\mathcal{G} = (\pi, L, m)$  be a holomorphic gerbe. Then we find arbitrarily fine open covers  $\{U_\alpha\}$  which admit holomorphic sections  $s_\alpha: U_\alpha \rightarrow Y$  of  $\pi$  and holomorphic trivializations  $\sigma_{\alpha\beta}$  of  $(s_\alpha, s_\beta)^*L$ .*

For such an open cover we define  $g_{\alpha\beta\gamma} \in \mathcal{O}^*(U_{\alpha\beta\gamma})$  by the equation

$$m(\sigma_{\alpha\beta}, \sigma_{\beta\gamma}) = g_{\alpha\beta\gamma} \cdot \sigma_{\alpha\gamma}.$$

Associativity of  $m$  implies that  $(g_{\alpha\beta\gamma})$  is closed. Different trivializations give cohomologous cocycles. Taking the limit over a cofinal sequence of covers, we define:

**Definition 9.** The *Dixmier-Douady class*  $\text{DD}(\mathcal{G}) \in \check{H}^3(X, \mathbb{Z}(1)_D)$  of the holomorphic gerbe  $\mathcal{G}$  on  $X$  is represented by the Čech cocycle  $(g_{\alpha\beta\gamma})$ .

The exponential map to  $H^3(X; \mathbb{Z})$  maps this class to the topological Dixmier-Douady class of  $\mathcal{G}$ , defined for example in [Mur96, Mur10].

If  $\mathcal{G}$  is equipped with a connection, this class may be refined to  $H^3(X, \mathbb{Z}(2)_D)$ . For this we write the connection on  $(s_\alpha, s_\beta)^*L$  as  $d + A_{\alpha\beta}$ . Since  $m$  preserves the connection we have  $d \log(g_{\alpha\beta\gamma}) = A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta}$ . Hence  $(g_{\alpha\beta\gamma}, A_{\alpha\beta}) \in \check{C}^3(\{U_\alpha\}, \mathbb{Z}(2)_D)$  is a Čech cocycle. If, in addition, we suppose  $\mathcal{G}$  to be equipped with a curving, then  $f_\beta - f_\alpha = dA_{\alpha\beta}$  for  $f_\alpha = s_\alpha^*f$ , by (1) and so  $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, f_\alpha)$  defines a class in  $H^3(X, \mathbb{Z}(3)_D)$ .

**1.4. Atiyah Class.** Contrasting the smooth case, holomorphic connections and curvings cannot always be found. This is measured by the following generalization of the *Atiyah class*  $A(L)$  for holomorphic line bundles. The short exact sequence of sheaves  $0 \rightarrow \Omega^p[-p-1] \rightarrow \mathbb{Z}(p+1)_D \rightarrow \mathbb{Z}(p)_D \rightarrow 0$  induces the following exact sequences, whose coboundary maps may be regarded as generalized Atiyah classes:

$$\begin{aligned} H^3(X, \mathbb{Z}(2)_D) &\longrightarrow H^3(X, \mathbb{Z}(1)_D) \xrightarrow{B} H^2(X, \Omega^1) \\ H^3(X, \mathbb{Z}(3)_D) &\longrightarrow H^3(X, \mathbb{Z}(2)_D) \xrightarrow{C} H^1(X, \Omega^2) \end{aligned}$$

**Proposition 10.** *The images of the classes  $B(\mathcal{G})$  and  $\text{DD}(\mathcal{G})$  in  $H^3(X; \mathbb{C})$  agree.*

*Proof.* From the definition of the codifferential, one sees that  $B$  maps the Čech cocycle  $g_{\alpha\beta\gamma} \in \mathcal{O}^*(U_{\alpha\beta\gamma})$  to the class of  $d \log g_{\alpha\beta\gamma}$  in  $H^2(A_{\text{cl}}^1)$ . Therefore the image of  $d \log g_{\alpha\beta\gamma}$  in  $H^3(X; \mathbb{C})$  is the image of the Dixmier-Douady class  $\delta(g_{\alpha\beta\gamma})$ .  $\square$

**Corollary 11.** *For a compact Kähler manifold  $X$  the topological Dixmier-Douady class  $\text{DD}(\mathcal{G}) \in H^3(X; \mathbb{Z})$  of a gerbe with holomorphic connection is torsion.*

**1.5. Morphisms of Gerbes.** There is a naive notion (which we do not present) of isomorphism with which two gerbes may have the same Dixmier-Douady class without being isomorphic. When wishing to emphasise this, we also call the morphisms below stable isomorphisms, in accordance with usual terminology.

**Definition 12.** Let  $\mathcal{G} = (Y, L, m)$ ,  $\mathcal{G}' = (Y', L', m')$  be holomorphic gerbes on  $X$ . A *morphism*  $F$  from  $\mathcal{G}$  to  $\mathcal{G}'$  is a triple  $((\varsigma, \varsigma'), R, \phi)$  consisting of a submersion  $(\varsigma, \varsigma') : Z \rightarrow Y \times_X Y'$ , line bundle  $R \rightarrow Z$ , and bundle isomorphism

$$\phi : (\varsigma^{[2]})^* L \otimes \text{pr}_2^* R \rightarrow \text{pr}_1^* R \otimes (\varsigma'^{[2]})^* L'.$$

Hence  $\phi$  gives maps  $L_{y_1, y_2} \otimes R_{z_2} \rightarrow R_{z_1} \otimes L'_{y'_1, y'_2}$  for  $(z_1, z_2) \in Z^{[2]}$ , where  $\varsigma(z_k) = y_k$ ,  $\varsigma'(z_k) = y'_k$ . For  $(z_1, z_2, z_3) \in Z^{[3]}$  we require commutative diagrams:

$$\begin{array}{ccc} L_{y_1, y_2} \otimes L_{y_2, y_3} \otimes R_{z_3} & \xrightarrow{\text{id} \otimes \phi} & L_{y_1, y_2} \otimes R_{z_2} \otimes L'_{y'_2, y'_3} \xrightarrow{\phi \otimes \text{id}} R_{z_1} \otimes L'_{y'_1, y'_2} \otimes L'_{y'_2, y'_3} \\ m \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes m' \\ L_{y_1, y_3} \otimes R_{z_3} & \xrightarrow{\phi} & R_{z_1} \otimes L'_{y'_1, y'_3} \end{array}$$

The *composite* of the morphism  $(Z, R, \phi) : \mathcal{G} \rightarrow \mathcal{G}'$  with  $(Z', R', \phi') : \mathcal{G}' \rightarrow \mathcal{G}''$  has the submersion  $Z \times_{Y'} Z' \rightarrow Y \times_X Y''$  and line bundle  $R \otimes R'$ .

**Definition 13.** A *connection* on a morphism  $F = (Z, R, \phi) : \mathcal{G} \rightarrow \mathcal{G}'$  of gerbes with connections is a holomorphic connection on  $R$  making  $\phi$  connection-preserving.

**Proposition 14.** *Given a morphism  $F : \mathcal{G} \rightarrow \mathcal{G}'$ , the Dixmier-Douady classes  $\text{DD}(\mathcal{G}) = \text{DD}(\mathcal{G}')$  in  $H^3(X; \mathbb{Z}(1)_D)$  agree. If the gerbes and the morphism  $F$  have connections, the classes in  $H^3(X; \mathbb{Z}(2)_D)$  agree.*

Any two morphisms  $(R, \phi), (\tilde{R}, \tilde{\phi}) : \mathcal{G} \rightarrow \mathcal{G}'$  differ by a line bundle on the base  $X$ : the maps  $\phi$  and  $\tilde{\phi}$  define descent data for the line bundle  $R_z \otimes \tilde{R}_z^*$  on  $Z$ . Hence it is the pullback of a bundle on  $X$ . Similarly with connections.

**1.6. Transformations.** We now come to the 2-morphisms of our bicategory.

**Definition 15.** Let  $F = (Z, R, \phi)$  and  $\tilde{F} = (\tilde{Z}, \tilde{R}, \tilde{\phi})$  be morphisms from  $\mathcal{G} = (Y, L, m)$  to  $\mathcal{G}' = (Y', L', m')$ . A *transformation*  $\alpha : F \Rightarrow \tilde{F}$  consists of

- (1) A submersion  $(\kappa, \tilde{\kappa}) : W \rightarrow Z \times_{Y \times_X Y'} \tilde{Z}$
- (2) A bundle isomorphism  $\psi : \kappa^* R \rightarrow \tilde{\kappa}^* \tilde{R}$  over  $W$ .

So  $\psi$  induces maps  $\psi_w: R_z \rightarrow \tilde{R}_{\tilde{z}}$ , for  $(\kappa, \tilde{\kappa})(w) = (z, \tilde{z})$ . For  $(w_1, w_2) \in W^{[2]}$  let  $(\kappa, \tilde{\kappa})(w_k) = (z_k, \tilde{z}_k)$ ,  $(\varsigma, \varsigma')(z_k) = (y_k, y'_k)$ . We require the commutativity of

$$\begin{array}{ccc} L_{y_1, y_2} \otimes R_{z_2} & \xrightarrow{\phi} & R_{z_1} \otimes L'_{y'_1, y'_2} \\ \text{id} \otimes \psi \downarrow & & \downarrow \psi \otimes \text{id} \\ L_{y_1, y_2} \otimes \tilde{R}_{\tilde{z}_2} & \xrightarrow[\phi]{} & \tilde{R}_{\tilde{z}_1} \otimes L'_{y'_1, y'_2} \end{array}$$

If the morphisms  $F, \tilde{F}$  are equipped with connections, the transformation  $\alpha$  is *compatible with the connections* if  $\psi$  preserves connections.

Two transformations are identified if they coincide on a pullback, see [Wal07]. This gives a bicategory of gerbes.

**1.7. Further Operations.** Let  $\mathcal{G} = (Y, L, m)$  be a gerbe on  $X$  and let  $f: X' \rightarrow X$  be a holomorphic map. The *pullback gerbe*  $f^*\mathcal{G}$  is given by  $Y' = Y \times_X X' \rightarrow X'$  and the pullback line bundle of  $L$  along  $(Y')^{[2]} \rightarrow Y^{[2]}$ .

The *tensor product*  $\mathcal{G} \otimes \mathcal{G}'$  of two gerbes  $\mathcal{G} = (Y, L, m)$ ,  $\mathcal{G}' = (Y', L', m')$  on  $X$  has the submersion  $Y \times_X Y' \rightarrow X$  and the exterior product line bundle  $L \otimes L'$ . Similarly, we have a tensor product of morphisms. For more details, see [Wal07].

The Dixmier-Douady class is compatible with pullback and additive with respect to tensor products. Equipping  $f^*\mathcal{G}$  and  $\mathcal{G} \otimes \mathcal{G}'$  with the tensor product and pullback connections, this also holds for the 3-curvature.

## 2. THE CANONICAL GERBE ON $\text{GL}_n(\mathbb{C})$

In this section, we give the details of the construction for the canonical gerbe  $\mathcal{G}^{\text{can}}$  on  $\text{GL}(n, \mathbb{C})$ . Theorem 1 asserts that  $\mathcal{G}^{\text{can}}$  is a representative for the generator of  $H^2(\text{GL}(n, \mathbb{C}), \mathcal{O}^\times)$ . Its proof is given in 2.2.

**2.1. Construction of  $\mathcal{G}^{\text{can}}$ .** The *sector* of radii  $0 \leq r < R \leq +\infty$  between  $z_1 = r_1 e^{i\varphi_1}$  and  $z_2 = r_2 e^{i\varphi_2}$ , with  $0 < \varphi_2 - \varphi_1 \leq 2\pi$  and  $r_1, r_2 > 0$ , is the set

$$S^{r, R}(z_1, z_2) = \{se^{i\varphi} \mid r \leq s \leq R, \varphi_1 \leq \varphi \leq \varphi_2\}.$$

We write  $S(z_1, z_2)$  if  $r = 0, R = +\infty$ .

Recall that if  $\chi_A(X) = \det(A - X \cdot \text{id}) = \prod_\lambda (X - \lambda)^{n_\lambda}$  is the characteristic polynomial of a matrix  $A$ , then  $\mathbb{C}^n$  may be decomposed into the  $n_\lambda$ -dimensional generalized eigenspaces  $V_\lambda(A) = \ker(A - \lambda \cdot \text{id})^{n_\lambda}$ . For a subset  $S \subset \mathbb{C}$  let

$$V_S(A) = \bigoplus \{V_\kappa(A) \mid \kappa \in S\}.$$

For  $\lambda, \mu \in \mathbb{C}$  and  $A \in \text{GL}_n(\mathbb{C})$  we introduce the notation

$$A(\lambda \rightarrow \mu) = \Lambda^{\text{top}}(V_{S(\lambda, \mu)}(A)).$$

By an eigendirection we mean a ray  $\mathbb{R}_{>0}\lambda$  containing an eigenvalue of  $A$ . By comparing sectors, the following is easy to show:

**Lemma 16.** *If  $\mu$  and at least one of  $\{\lambda, \kappa\}$  are not eigendirections of  $A$  we have*

$$A(\lambda \rightarrow \mu) \otimes A(\mu \rightarrow \kappa) = A(\lambda \rightarrow \kappa). \quad (2)$$

Moreover, for  $\lambda$  not an eigendirection we have

$$A(\lambda \rightarrow \lambda) = \mathbb{C}, \quad (3)$$

$$A(\lambda \rightarrow \mu) \cong A(\mu \rightarrow \lambda)^*. \quad (4)$$

We now define the projection  $\pi: Y \rightarrow \mathrm{GL}_n(\mathbb{C})$  of the gerbe  $\mathcal{G}^{\mathrm{can}}$ . Let

$$Y = \{(A, z) \in \mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}^\times \mid \forall r > 0 : \det(A - rz \cdot \mathrm{id}) \neq 0\}.$$

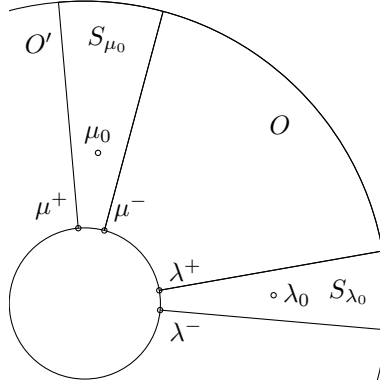
Using that all eigenvalues are bounded by the operator norm of a matrix, one shows the set  $\{(A, \lambda) \in \mathrm{GL}_n(\mathbb{C}) \times S^1 \mid \text{the ray } \mathbb{R}_{>0}\lambda \text{ contains an eigenvalue of } A\}$  of eigendirections to be closed. It follows that  $Y$  is an open subset of  $\mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}^\times$ , so it is a complex manifold. Clearly, the projection  $\pi: Y \rightarrow \mathrm{GL}_n(\mathbb{C})$  is a submersion.

The line bundle  $L \rightarrow Y^{[2]}$  of the gerbe  $\mathcal{G}^{\mathrm{can}}$  is given by the determinant lines

$$L_{(A, \lambda, \mu)} = A(\lambda \rightarrow \mu).$$

In particular,  $L_{(A, \lambda, \mu)}$  depends only on  $\mathbb{R}_{>0}\lambda$  and  $\mathbb{R}_{>0}\mu$ . The multiplication  $m$  in the gerbe is given by (2). It is associative since the identifications (2) are associative.

**2.2. Proof of Theorem 1.** We prove that  $L$  is holomorphic near each point  $(A_0, \lambda_0, \mu_0)$ . Suppose first that  $\lambda_0 \neq \mu_0$ . Pick real numbers  $0 < r < R$  so that the spectrum of  $A_0$  is contained in the annulus of radii  $r, R$ . Since the spectrum of  $A_0$  is finite we find disjoint sectors  $S_{\mu_0} = S^{r, R}(\mu^-, \mu^+)$ ,  $S_{\lambda_0} = S^{r, R}(\lambda^-, \lambda^+)$  with non-empty interior and containing  $\mu_0, \lambda_0$  respectively, but no eigenvalues of  $A_0$ . Let  $O = S^{r, R}(\lambda^+, \mu^-)$ ,  $O' = S^{r, R}(\mu^+, \lambda^-)$  be the complementary sectors.



Consider the lower semi-continuous function  $\varphi(A) = \sup |\det(A - \lambda E_n)|$  on  $\mathrm{GL}_n(\mathbb{C})$ , the supremum ranging over  $\lambda \in \partial O \cup \partial O' \cup \partial S_{\mu_0} \cup \partial S_{\lambda_0}$ . As  $\varphi(A_0) > 0$  we find a connected neighbourhood  $U \subset \{A \mid \varphi(A) > \varphi(A_0)/2\}$  of  $A_0$ . The zero counting integral is a holomorphic  $\mathbb{Z}$ -valued function of  $A$  and therefore constant on  $U$ :

$$\frac{1}{2\pi i} \oint \frac{\chi'_A(\lambda)}{\chi_A(\lambda)} d\lambda \quad (5)$$

Applied to the boundary paths of  $O, O', S_{\mu_0}, S_{\lambda_0}$  we see that  $A$  has no eigenvalues in  $S_{\lambda_0}$  or  $S_{\mu_0}$  and also that all eigenvalues remain within  $O \cup O'$ .

It follows that for  $(A, \lambda, \mu) \in U \times S_{\lambda_0} \times S_{\mu_0}$  the line  $L_{(A, \lambda, \mu)}$  may be identified with the top exterior power of  $V_O(A)$ . It suffices now to prove that  $V_O \rightarrow U$  is a holomorphic sub vector bundle of the trivial bundle. Fix  $A_1 \in U$ .

We have holomorphic maps  $e, e'$  on  $O \cup O'$  defined by  $e|_O = 1$ ,  $e|_{O'} = 0$  and  $e' = 1 - e$ . Holomorphic functional calculus gives us spectral idempotents  $e_A, e'_A \in M_n(\mathbb{C})$  depending holomorphically on  $A \in U$ . It is well-known that  $e_A$  and  $e'_A$  are the complex-linear projection maps onto the generalized eigenspaces  $V_O$  and  $V_{O'}$ . We regard  $e(A) = e_A$  as a bundle endomorphism of  $U \times \mathbb{C}^n$ . The subspaces  $\ker e_{A_1}$  and  $\operatorname{im} e_{A_1}$  are complementary in  $\mathbb{C}^n$ . It follows that the holomorphic map

$$\psi = e \oplus \operatorname{incl}: (U \times \operatorname{im} e_{A_1}) \oplus (U \times \ker e_{A_1}) \rightarrow U \times \mathbb{C}^n$$

is the identity on the fiber over  $A_1$ . The map  $\psi$  remains invertible on fibers close to  $A_1$ , so by restricting we get a holomorphic isomorphism from the trivial bundle  $U \times \operatorname{im} e_{A_1}$  to  $\psi(U \times \operatorname{im} e_{A_1})$ . Applying (5) to  $\partial O$  shows that the fiber dimensions in  $\psi(U \times \operatorname{im} e_{A_1}) \subset \operatorname{im}(e)$  coincide, so we have equality. That is,  $\psi|_{U \times \operatorname{im} e_{A_1}}$  is a trivialization of  $\operatorname{im}(e) = V_O$  near  $A_1$ .

If  $\lambda_0 = \mu_0$  we proceed similarly: choose a small sector  $S$  around  $\lambda_0$  containing no eigenvalues of  $A_0$ . Applying the above argument gives a neighbourhood  $U$  of  $A_0$  so that no  $A \in U$  has an eigenvalue in  $S$ . Hence  $L|_{U \times S \times S}$  is the trivial line bundle.

We now show that  $m$  is holomorphic in a neighbourhood of  $(A_0, \lambda_0, \mu_0, \kappa_0)$ . For this we first suppose that the directions  $\mathbb{R}_{>0}\lambda_0, \mathbb{R}_{>0}\mu_0, \mathbb{R}_{>0}\kappa_0$  are pairwise different. Choose disjoint sectors  $\lambda_0 \in S_{\lambda_0} = S^{r,R}(\lambda^-, \lambda^+)$ ,  $\mu_0 \in S_{\mu_0} = S^{r,R}(\mu^-, \mu^+)$ ,  $\kappa_0 \in S_{\kappa_0} = S^{r,R}(\kappa^-, \kappa^+)$  with non-empty interior and containing no eigenvalues of  $A_0$ . Letting  $O = S^{r,R}(\lambda^+, \mu^-)$ ,  $O' = S^{r,R}(\mu^+, \kappa^-)$ , and  $O'' = S^{r,R}(\kappa^+, \lambda^-)$  we find as above a neighbourhood  $U$  of  $A_0$  in which

$$L_{(A,\lambda,\mu)} = \Lambda^{\operatorname{top}}(V_O), \quad L_{(A,\mu,\kappa)} = \Lambda^{\operatorname{top}}(V_{O'}), \quad L_{(A,\lambda,\kappa)} = \Lambda^{\operatorname{top}}(V_{O''})$$

whenever  $(A, \lambda, \mu, \kappa) \in U \times S_{\lambda_0} \times S_{\mu_0} \times S_{\kappa_0}$ . There are two cases:  $\arg(\kappa_0/\mu_0) + \arg(\mu_0/\lambda_0) < 2\pi$  and  $> 2\pi$ . Depending on the case, the map  $m$  is given on this neighbourhood by the top exterior power of either the addition map  $V_O \oplus V_{O'} \rightarrow V_{O''}$  or of  $V_O \oplus V_{O'} \rightarrow V_{\mathbb{C}} \oplus V_{O''} \rightarrow V_{O''}$ , both of which are holomorphic. The cases where some of the directions  $\lambda_0, \mu_0, \kappa_0$  agree are even simpler.

The restriction of  $\mathcal{G}_{\text{can}}$  to  $U(n)$  is the tautological smooth gerbe of Murray-Stevenson [MS08]. Therefore the Dixmier-Douady class of  $\mathcal{G}_{\text{can}}$  is the generator of  $H^3(\operatorname{GL}_n(\mathbb{C}); \mathbb{Z})$ , by functoriality. This will be improved in Corollary 19 below, completing the proof of Theorem 1.

### 3. COHOMOLOGICAL THEORY ON STEIN MANIFOLDS

In this section we collect a number of facts for the Deligne cohomology of Stein manifolds. These will be needed in the proof of Theorem 2 in the next section.

**3.1. Stein manifolds.** We recall that a domain  $U \subset \mathbb{C}^n$  is a Stein manifold if for every boundary point  $z \in \partial U$  there exists a holomorphic function on  $U$  that cannot be continued analytically across  $z$  (see [GR04b]).

**Definition 17.** A complex Lie group  $G$  is a *Stein group* if the underlying manifold is a Stein manifold (see [GR04b, p. 136]).

$\operatorname{GL}(n, \mathbb{C})$  and any closed complex subgroup of  $\operatorname{GL}(n, \mathbb{C})$  is a Stein group. Any semi-simple connected or simply-connected solvable complex Lie group is Stein.



**Proposition 18.** *Let  $X$  be a contractible Stein manifold (for example, a polycylinder) and let  $\mathcal{G}, \mathcal{G}'$  be gerbes on  $X$ . Then any two stable isomorphisms are isomorphic, meaning that we find a transformation between them.*

*Proof.* Two stable morphisms differ by a line bundle on  $X$ , which in our case is a holomorphic line bundle  $L$ . But  $L$  is holomorphically trivial, by the Grauert-Oka Theorem [Gra58], and a trivialization defines a transformation.  $\square$

**3.2. Exponential Sequence.** The long exact sequence induced by

$$0 \rightarrow \Omega_X^{\bullet < p}[-1] \rightarrow \mathbb{Z}(p)_D \rightarrow \mathbb{Z} \rightarrow 0$$

with the fact  $H^*(\Omega^{< p}) = H^*(X; \mathbb{C})$  for  $* < p - 1$  shows that  $H^q(\mathbb{Z}(p)_D)$  sits inside a Bockstein sequence. By the Five Lemma we get

$$H^q(X; \mathbb{Z}(p)_D) \cong H^{q-1}(X; \mathbb{C}/\mathbb{Z}), \quad 0 < q < p - 1 \quad (6)$$

The case  $q \geq p$  is more difficult and leads to the *exponential sequence*

$$\rightarrow H^{q-1}(X; \mathbb{Z}) \rightarrow H^{q-1}(\Omega^{< p}) \rightarrow H^q(\mathbb{Z}(p)_D) \rightarrow H^q(X; \mathbb{Z}) \rightarrow H^q(\Omega^{< p}) \rightarrow$$

If  $X$  is a Stein manifold and  $q > p > 0$  then  $H^q(\Omega^{< p}) = 0$ , by Cartan's Theorem B [Car53, p. 51]. Putting this into the exponential sequence gives

$$H^q(\mathbb{Z}(p)_D) \cong H^q(X; \mathbb{Z}), \quad q > p. \quad (7)$$

**Corollary 19.** *The class  $\text{DD}(\mathcal{G}_{\text{can}})$  of the canonical gerbe is the generator of  $H^3(\text{GL}(n, \mathbb{C}), \mathbb{Z}(1)_D) \cong H^3(\text{GL}(n, \mathbb{C}), \mathbb{Z}) = \mathbb{Z}$ .*

Since  $H^3(\text{GL}(n, \mathbb{C}), \mathbb{Z}(2)_D) \cong H^3(\text{GL}(n, \mathbb{C}), \mathbb{Z}(1)_D)$  there is a unique holomorphic connection on  $\mathcal{G}_{\text{can}}$ , up to stable isomorphisms with connection. This connection may also be constructed explicitly by  $\mathbb{C}$ -linear projection  $\mathbb{C}^n \rightarrow V_O(A)$  onto the eigenspace bundles.

#### 4. MULTIPLICATIVE STRUCTURE

The existence of a multiplicative structure depends only on the stable isomorphism class of the gerbe and is therefore a cohomological problem. After the proof of existence, we describe the multiplicative structure more explicitly.

**Definition 20.** Let  $G$  be complex Lie group with product  $\mu$ . A *multiplicative holomorphic gerbe* on  $G$  consists of a holomorphic gerbe  $\mathcal{G}$  on  $G$ , a morphism  $M: \text{pr}_1^* \mathcal{G} \otimes \text{pr}_2^* \mathcal{G} \rightarrow \mu^* \mathcal{G}$ , and a *transformation*  $\alpha: M \circ (M \otimes \text{id}) \Rightarrow M \circ (\text{id} \otimes M)$ . The transformation  $\alpha$  should fit into the usual coherence pentagon [Wal10, p. 47]. A *connection* on a multiplicative gerbe consists of connections on  $\mathcal{G}$  and  $M$  so that  $\alpha$  is compatible with the connections.

In general, there are obstructions to finding a multiplicative structure on a given gerbe  $\mathcal{G}$ . For Stein groups, Theorem 2, which we shall prove in this section, asserts that this obstruction against a *holomorphic* multiplicative structure reduces to a *topological* problem.

Recall from [Del74] that a simplicial space is a functor  $X: \Delta^{\text{op}} \rightarrow \mathbf{Top}$ . For example, the constant simplicial space has  $X_n = X$  for a fixed space  $X$  and all faces and degeneracies are the identity. Next, for a topological group  $G$  we let  $BG_\bullet$  denote the nerve of  $G$  with  $n$ -simplices  $BG_n = G^n$ . One finds a simplicial map  $EG_\bullet \rightarrow BG_\bullet$  whose fibers are the constant simplicial spaces  $G$ , see [Seg68].

The cohomology of simplicial space  $X_\bullet$  with coefficients in a sheaf  $\mathcal{F}$  is the cohomology of the total complex of the double Čech complex  $C^{pq} = \check{C}^q(X_p, \mathcal{F})$  (one may also use injective resolutions). Reduced cohomology is defined as usual way by passing to quotient complexes. On constant simplicial spaces one recovers usual sheaf cohomology. Similarly, for a complex of sheaves  $\mathcal{F}^*$  one defines  $H(X_\bullet, \mathcal{F}^*)$  using the Čech hypercomplex [Bry08, p. 28]. From [Seg68] we take that the simplicial cohomology  $H(BG_\bullet; \mathbb{Z})$  computes the cohomology of the classifying space  $BG = |BG_\bullet|$  of a Lie group  $G$ .

For a double complex  $F^p$  denotes the  $p$ -th vertical filtration  $C^{iq}, i \geq p$ . We have a short exact sequence of Čech cochain complexes

$$0 \rightarrow F^2 \rightarrow \check{C}^*(BG_\bullet, \text{pt}; \mathcal{F}) \xrightarrow{\tau} \frac{\check{C}^*(BG_\bullet, \text{pt}; \mathcal{F})}{F^2} \rightarrow 0$$

Identifying the rightmost term with  $\check{C}^{*-1}(G, \mathcal{F})$ , the map  $\tau$  simply collapses all but the second column of the double complex. In cohomology  $\tau$  induces the transgression map, see [Wal10, Lemma 2.9]. Using Lemma 7, a gerbe with holomorphic connection may be described, up to isomorphism, by a cocycle in  $\check{C}^3(G, \mathbb{Z}(2)_D)$ . The data for the morphism  $M$  and transformation  $\alpha$  in Definition 20 corresponds exactly to an extension to a cocycle of the double complex  $\check{C}^4(BG_\bullet, \text{pt}; \mathbb{Z}(2)_D)$ .

To prove Theorem 2 it remains therefore only to show:

**Lemma 21.** *The map  $H^*(BG_\bullet, \mathbb{Z}(2)_D) \rightarrow H^*(BG; \mathbb{Z})$  is an isomorphism ( $*$   $> 0$ ).*

*Proof.* By the long exact sequence induced by  $0 \rightarrow \Omega^{<2}[-1] \rightarrow \mathbb{Z}(2)_D \rightarrow \mathbb{Z} \rightarrow 0$  it suffices to show  $H^*(BG_\bullet, \Omega^{<2}) = 0$ . To do this, we consider the spectral sequence for simplicial spaces  $E_1^{pq} = H^q(BG_p, \Omega^{<2}) \Rightarrow H^{p+q}(BG_\bullet; \Omega^{<2})$ .

From Cartan's Theorem B [Car53, p. 51] and the sheaf hypercohomology spectral sequence we deduce for any Stein manifold  $X$  that the space  $H^*(X, \Omega^{<2})$  is  $H^0(X; \mathbb{C})$  for  $*$   $= 0$ ,  $H^1(X; \mathbb{C})$  for  $*$   $= 1$ , and zero else.

It follows that  $E_1^{pq} = 0$  unless  $q = 0, 1$ . The first row  $E_1^{p0} = H^0(G^p; \mathbb{C})$  is given by  $\mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C} \rightarrow \dots$  and in particular exact apart from the first term. The second row  $E_1^{p1}$  reads  $0 \rightarrow H^1(G; \mathbb{C}) \rightarrow H^1(G^2; \mathbb{C}) \rightarrow H^1(G^3; \mathbb{C}) \rightarrow \dots$ , where the maps are  $\sum_i (-1)^i d_i^*$  induced by the face maps  $d_i$  of the nerve. Since we are in degree one,  $\mu^*: H^1(G; \mathbb{C}) \rightarrow H^1(G^2; \mathbb{C})$  equals  $\text{pr}_1^* + \text{pr}_2^*$ , from which it follows that the row  $E_1^{p1}$  is exact. Hence  $E_2^{pq} = 0$  for  $p + q > 0$ , whence the result.  $\square$

**Definition 22.** A choice of preimage of  $\text{DD}(\mathcal{G}) \in H^3(G; \mathbb{Z}(2)_D)$  is the *multiplicative class*  $\lambda(\mathcal{G}) \in H^4(BG_\bullet; \mathbb{Z}(2)_D)$  of the multiplicative gerbe with holomorphic connection (see [Wal10] in the smooth category).

From the proof above, it is clear that the multiplicative class determines the multiplicative structure, up to (multiplicative) isomorphism, see [Wal10].

**Corollary 23.** *For the group  $G = \text{GL}(n, \mathbb{C})$  every holomorphic gerbe with connection admits a multiplicative structure.*

*Proof.* From Lemma 21 we deduce

$$H^4(B\text{GL}(n, \mathbb{C})_\bullet, \mathbb{Z}(2)_D) \cong H^4(B\text{GL}(n, \mathbb{C}); \mathbb{Z}) = \mathbb{Z}c_1^2 \oplus \mathbb{Z}c_2.$$

The preimages  $c_1^B \in H^2(BG_\bullet; \mathbb{Z}(1)_D)$ ,  $c_2^B \in H^4(BG_\bullet; \mathbb{Z}(2)_D)$  in Deligne cohomology of these classes are the *Beilinson-Chern classes*. On the other hand, by (7)

$$H^3(G; \mathbb{Z}(2)_D) \cong H^3(G; \mathbb{Z}) = \mathbb{Z}.$$

The topological transgression map takes  $c_2$  to the generator of  $\mathbb{Z}$  (see [Bor53]). It follows that every class  $[\mathcal{G}] \in H^3(G; \mathbb{Z}(2)_D)$  is in the image of  $\tau$ .  $\square$

**4.1. Explicit Description of Multiplicative Structure.** For  $\rho \in \mathbb{C}^\times$  let  $U_\rho$  be the subset of  $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$  of all matrices  $(A, B)$  having the property that neither of  $A, B, A \cdot B$  have an eigenvalue on the ray  $\mathbb{R}_{>0}\rho$ . It is not hard to see that the set  $U_\rho$  is an open dense subset of  $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$  whose connected components are Stein manifolds. Unfortunately, the homotopy type of  $U_\rho$  is not known to the author. Therefore let  $P \subset U_\rho$  be polycylinder. Over  $P$  we have an obvious stable morphism  $\mathrm{pr}_1^* \mathcal{G} \otimes \mathrm{pr}_2^* \mathcal{G} \rightarrow \mu^* \mathcal{G}$  defined as follows. It consists of a submersion  $\varsigma$  to the space  $Z$  of all  $z = (A, B, \lambda, \mu, \kappa)$  such that  $\lambda$  is no eigendirection of  $A$ ,  $\mu$  is no eigendirection of  $B$ , and  $\kappa$  is no eigendirection of  $AB$ . We choose  $\varsigma = \mathrm{id}_Z$ . Let the line bundle  $R \rightarrow Z$  over  $z = (A, B)$  be given by

$$R_z = A(\lambda \rightarrow \rho) \otimes B(\mu \rightarrow \rho) \otimes (AB)(\rho \rightarrow \kappa).$$

For this choice of  $R$  both sides of

$$\psi: L_{(A, \lambda_1, \lambda_2)} \otimes L_{(B, \mu_1, \mu_2)} \otimes R_{z_2} \rightarrow R_{z_1} \otimes L_{AB, \kappa_1, \kappa_2}$$

are identical, by (2). As isomorphism  $\psi$  (appearing in the definition of morphism) we may therefore use the identity. By Proposition 18, this explicit morphism and the one abstractly given by Theorem 2 are isomorphic. For the explicit morphism there are also obvious associativity transformations

$$R_{(A_1, A_2, \lambda_1, \lambda_2, \lambda_{12})} \otimes R_{(A_1 A_2, A_3, \lambda_{12}, \lambda_3, \lambda_{123})} \rightarrow R_{(A_1, A_2 A_3, \lambda_1, \lambda_{23}, \lambda_{123})} \otimes R_{(A_2, A_3, \lambda_2, \lambda_3, \lambda_{23})},$$

where  $\lambda_{ijk\dots}$  is a non-eigendirection of  $A_{ijk\dots}$ . Using (4), we get a canonical identification  $\psi$  of these two line bundles. The commutativity of the pentagon (9) is straight-forward, using the associativity of  $\psi$ .

## 5. HOLOMORPHIC 2-GERBES

**5.1. 2-Gerbes.** Our definition of 2-gerbe is weaker than that in [Ste04].

**Definition 24.** A *holomorphic 2-gerbe*  $\mathfrak{G} = (\rho, \mathcal{G}, M, \alpha)$  on  $X$  consists of

- (1) A submersion  $\rho: V \rightarrow X$ .
- (2) A holomorphic gerbe  $\mathcal{G}$  on  $V^{[2]}$ .
- (3) A morphism of gerbes  $M: \mathrm{pr}_{12}^* \mathcal{G} \otimes \mathrm{pr}_{23}^* \mathcal{G} \rightarrow \mathrm{pr}_{13}^* \mathcal{G}$  over  $V^{[3]}$ .
- (4) A transformation  $\alpha: M \circ (M \otimes \mathrm{id}) \Rightarrow M \circ (\mathrm{id} \otimes M)$  between morphisms  $\mathrm{pr}_{12}^* \mathcal{G} \otimes \mathrm{pr}_{23}^* \mathcal{G} \otimes \mathrm{pr}_{34}^* \mathcal{G} \rightarrow \mathrm{pr}_{14}^* \mathcal{G}$ .

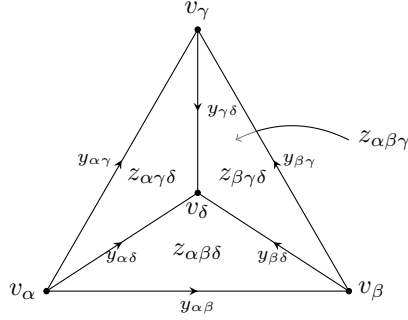
The transformations  $\alpha$  are required to fit into the usual commutative pentagon [Wal10, p. 47]. In detail, this means the commutativity of diagram (9) below.

**Remark 25.** Let us unwind this definition and see that it entails an  $X$ -indexed family of 2-categories. The elements  $v \in \rho^{-1}(x)$  are the *objects* at  $x \in X$ . The gerbe  $\mathcal{G}$  contains a submersion  $\pi = (\pi_1, \pi_2): Y \rightarrow V^{[2]}$  whose fibers  $y \in \pi^{-1}(v_1, v_2)$  are the *1-arrows* from  $v_1$  to  $v_2$ . Given two 1-arrows  $y_1, y_2$  we have a complex line  $L_{y_1, y_2}$  of 2-arrows. The multiplication in the gerbe  $\mathcal{G}$  gives a strictly associative *vertical composition* of 2-arrows. Horizontal composition is encoded in the morphism  $M$ .

It includes a submersion  $Z \rightarrow (Y \times_{\pi_1} Y) \times_{V[2]} Y$  onto the set of composable triangles of 1-arrows. We regard  $z \in Z$  mapping to  $(y_{\alpha\beta}, y_{\beta\gamma}, y_{\alpha\gamma})$  as a *filler* of this triangle and then think of  $y_{\alpha\gamma}$  as a choice of horizontal composition of  $y_{\alpha\beta}$  and  $y_{\beta\gamma}$ . In particular, it is not unique. Given two filled triangles  $z \mapsto (y_{\alpha\beta}, y_{\beta\gamma}, y_{\alpha\gamma})$  and  $\tilde{z} \mapsto (\tilde{y}_{\alpha\beta}, \tilde{y}_{\beta\gamma}, \tilde{y}_{\alpha\gamma})$  on the same vertices,  $M$  provides us with isomorphisms

$$L_{y_{\alpha\beta}, \tilde{y}_{\alpha\beta}} \otimes L_{y_{\beta\gamma}, \tilde{y}_{\beta\gamma}} \otimes R_{\tilde{z}} \rightarrow R_z \otimes L_{y_{\alpha\gamma}, \tilde{y}_{\alpha\gamma}}.$$

This is the horizontal composition of 2-arrows. The morphism  $\alpha$  includes a submersion  $W \rightarrow (Z \times_Y Z) \times_{Y[4]} (Z \times_Y Z)$  to fillers  $(z_{\beta\gamma\delta}, z_{\alpha\gamma\delta}, z_{\alpha\beta\delta}, z_{\alpha\beta\gamma})$  of diagrams:



We regard a preimage  $w_{\alpha\beta\gamma\delta}$  as a filler of this tetrahedron.  $\alpha$  gives an isomorphism

$$\psi_{w_{\alpha\beta\gamma\delta}} : R_{z_{\alpha\beta\delta}} \otimes R_{z_{\beta\gamma\delta}} \rightarrow R_{z_{\alpha\gamma\delta}} \otimes R_{z_{\alpha\beta\gamma}}. \quad (8)$$

These mediate between the two ways to horizontally compose three 2-arrows. The commutative pentagon mentioned above is then expressed as follows. Let  $\alpha, \beta, \gamma, \delta, \varepsilon$  be objects with arrows  $y_{\alpha\beta}, \dots, y_{\delta\varepsilon}$  between them, let  $z_{\alpha\beta\gamma}, \dots, z_{\gamma\delta\varepsilon}$  be fillers of all triangles, and let  $w_{\beta\gamma\delta\varepsilon}, w_{\alpha\gamma\delta\varepsilon}, w_{\alpha\beta\delta\varepsilon}, w_{\alpha\beta\gamma\varepsilon}, w_{\alpha\beta\gamma\delta}$  be fillers of the resulting tetrahedra. For every such data, we have a commutative diagram:

$$\begin{array}{ccc} & R_{\alpha\beta\gamma} \otimes R_{\alpha\gamma\delta} \otimes R_{\alpha\delta\varepsilon} & \\ \psi_{\alpha\beta\gamma\delta} \otimes \text{id} \swarrow & & \searrow \text{id} \otimes \psi_{\alpha\gamma\delta\varepsilon} \\ R_{\beta\gamma\delta} \otimes R_{\alpha\beta\delta} \otimes R_{\alpha\delta\varepsilon} & & R_{\alpha\beta\gamma} \otimes R_{\gamma\delta\varepsilon} \otimes R_{\alpha\gamma\varepsilon} \\ \text{id} \otimes \psi_{\alpha\beta\delta\varepsilon} \downarrow & & \downarrow \text{id} \otimes \psi_{\alpha\beta\gamma\varepsilon} \\ R_{\beta\gamma\delta} \otimes R_{\beta\delta\varepsilon} \otimes R_{\alpha\beta\varepsilon} & \xrightarrow{\psi_{\beta\gamma\delta\varepsilon} \otimes \text{id}} & R_{\beta\gamma\varepsilon} \otimes R_{\gamma\delta\varepsilon} \otimes R_{\alpha\beta\varepsilon} \end{array} \quad (9)$$

**5.2. The Dixmier-Douady Class.** Let  $\mathfrak{G} = (\rho, \mathcal{G}, M, \alpha)$  be a 2-gerbe on  $X$ . By Lemma 7 we find open covers  $\{U_\alpha\}$  of  $X$  with holomorphic sections as follows:

- (1)  $v_\alpha : U_\alpha \rightarrow V$  of  $\rho$ .
- (2)  $y_{\alpha\beta} : U_{\alpha\beta} \rightarrow Y$  of the pullback of  $\pi$  along  $(v_\alpha, v_\beta) : U_{\alpha\beta} \rightarrow V[2]$ .
- (3)  $z_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \rightarrow Z$  of the pullback of  $Z \rightarrow (Y \times_V Y) \times_{V[2]} Y$  along  $(y_{\alpha\beta}, y_{\beta\gamma}, y_{\alpha\gamma})$ .
- (4)  $w_{\alpha\beta\gamma\delta} : U_{\alpha\beta\gamma\delta} \rightarrow W$  of the pullback of  $W \rightarrow (Z \times_Y Z) \times_{Y[4]} (Z \times_Y Z)$  along  $(z_{\beta\gamma\delta}, z_{\alpha\beta\delta}, z_{\alpha\beta\gamma}, z_{\alpha\gamma\delta}) : U_{\alpha\beta\gamma\delta} \rightarrow Z^4$ .
- (5) Trivializations of  $z_{\alpha\beta\gamma}^* R$

Since both sides are trivialized, (8) is just a holomorphic function  $g_{\alpha\beta\gamma\delta} \in \mathcal{O}^*(U_{\alpha\beta\gamma\delta})$ .

**Definition 26.** The *Dixmier-Douady class* of  $\mathfrak{G}$  is  $[(g_{\alpha\beta\gamma\delta})] = \text{DD}(\mathfrak{G}) \in \check{H}^4(X, \mathbb{Z}(1)_D)$ .

**Definition 27.** Let  $\mathfrak{G} = (\rho, \mathcal{G}, M, \alpha)$  be a holomorphic 2-gerbe. A *connection* on  $\mathfrak{G}$  consists of a connection  $\nabla^L$  on the gerbe  $\mathcal{G}$  and a connection  $\nabla^R$  on the morphism  $M$ . The transformation  $\alpha$  is required to be compatible with the connections.

Write the connection on  $z_{\alpha\beta\gamma}^* R$  as  $d + A_{\alpha\beta\gamma}$ . Since (8) preserves connections

$$d \log g_{\alpha\beta\gamma\delta} = A_{\alpha\beta\delta} + A_{\beta\gamma\delta} - A_{\alpha\gamma\delta} - A_{\alpha\beta\gamma}.$$

Thus  $(g_{\alpha\beta\gamma\delta}, A_{\alpha\beta\gamma})$  refines the Dixmier-Douady class to  $H^4(X, \mathbb{Z}(2)_D)$ .

**5.3. The Canonical 2-Gerbe.** For a holomorphic vector bundle  $E$  over an algebraic manifold  $X$  we have the Beilinson-Chern classes (see [Beĩ84, Bry99a, EV88])

$$c_p^B(E) \in H^{2p}(X, \mathbb{Z}(p)_D).$$

By [EV88, Proposition 8.2] these are characterized by functoriality and by the requirement that they map to the Chern classes via  $H^{2p}(X, \mathbb{Z}(p)_D) \rightarrow H^{2p}(X; \mathbb{Z})$ .

The construction of such classes is based on Deligne's theory of mixed Hodge structures [Del74]. Our Lemma 21 above shows that for  $p \leq 2$  one may also deduce the existence of these classes from the theory of Stein spaces.

Recall from Theorem 1 the canonical gerbe  $\mathcal{G}^{\text{can}}$  on  $\text{GL}(n, \mathbb{C})$ . Using Corollary 23 we equip this gerbe with the multiplicative structure with multiplicative class  $c_2$ .

**Definition 28.** Let  $E \rightarrow X$  be a holomorphic vector bundle. Its *associated 2-gerbe*  $\mathfrak{G}(E)$  is defined by the following data:

- (1) As submersion  $\rho$  we take the principal bundle of frames  $P_{\text{GL}}(E) \rightarrow X$ .
- (2) The gerbe  $\mathcal{G}(E) = \delta^* \mathcal{G}^{\text{can}}$  with  $\delta: P_{\text{GL}}^{[2]} \rightarrow \text{GL}(n, \mathbb{C})$  given by  $p\delta(p, q) = q$ .
- (3)  $M$  and  $\alpha$  are pulled back from the multiplicative structure on  $\mathcal{G}^{\text{can}}$ .

For  $E = T^*X$  we call  $\mathfrak{G}(T^*X)$  the *canonical 2-gerbe* of the complex manifold.

*Proof of Theorem 3.* Functoriality is obvious. By the definition of the topological Dixmier-Douady class of a 2-gerbe one sees easily that it is the pullback of the topological multiplicative class  $\lambda(\mathcal{G}^{\text{can}}) = c_2 \in H^4(B\text{GL}(n, \mathbb{C}); \mathbb{Z})$  under the classifying map  $X \rightarrow B\text{GL}(n, \mathbb{C})$ . Alternatively, one may appeal to the smooth case [Wal10]. This proves (2). Now (3) follows from (1), (2) and [EV88, Proposition 8.2].  $\square$

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